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Mini Review

A tractroid realization of a 2d black hole vacuum

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Abstract

The two-dimensional black hole vacuum obtained from a spatial slice of the BTZ black hole is mapped explicitly to a tractroid surface minus a bounding circle.

Introduction

At a fixed time τ (for example $\tau = 0$) the 3d Euclidean BTZ black hole B_M [1,2] of mass $M > 0$ reduces to a 2d spatial slice whose metric ds_0^2 is easily transformed to a Poincare metric on the upper half-plane

$$H^+ \overset{def.}{=} \{(x,y) \in \mathbb{R}^2 \mid y > 0\}. \tag{1}$$

Moreover, the quotient $X_\Gamma \overset{def.}{=} \Gamma \backslash H^+$ of H^+ by a subgroup Γ of $G = SL(2, \mathbb{R})$ generated by a *parabolic* element γ (ie. trace $\gamma = \pm 2$) has for $M = 0$ the structure of a 2d black hole vacuum [3]. We indicate a realization of this vacuum by way of an explicit bijection $\tilde{\Phi} : T_a^+ \to X_\Gamma$, where T_a^+ is a tractroid surface with a deleted boundary circle of radius a.

The spatial slice of B_M

B_M, with zero angular momentum, is given by the metric with periodicity in the Schwarzschild variable ϕ

$$ds^2 = \left(\frac{r^2}{\ell^2} - M\right)d\tau^2 + \left(\frac{r^2}{\ell^2} - M\right)^{-1} dr^2 + r^2 d\phi^2. \tag{2}$$

ds^2 solves the Einstein vacuum field equations

$$R_{ij} - \frac{1}{2} R g_{ij} - \Lambda g_{ij} = 0 \tag{3}$$

with *negative* cosmological constant $\Lambda \overset{def.}{=} -1/\ell^2$, where ℓ in (2) is a positive constant. By our sign convention, the Ricci scalar curvature R in (3) is given by $R = 6/\ell^2$. ds_0^2 in the introduction is therefore given by

$$ds_0^2 \overset{def.}{=} \frac{dr^2}{\frac{r^2}{\ell^2} - M} + r^2 d\phi^2 \tag{4}$$

which by way of the transformation of variables

$$x = \phi, y = \ell / r > 0 \tag{5}$$

in case $M = 0$ reduces to the Poincare metric

$$ds_P^2 \overset{def.}{=} \ell^2 \left(\frac{dx^2 + dy^2}{y^2}\right) \tag{6}$$

on H^+ in (1). Specially for X_Γ, we choose

$$\Gamma \overset{def.}{=} \left\{ \begin{bmatrix} 1 & 2\pi n \\ 0 & 1 \end{bmatrix} \middle| \, n \in \mathbb{Z} \right\} = \left\{ \gamma^n \mid n \in \mathbb{Z} \right\} \tag{7}$$

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for $\mathbb{Z}$ = set of whole numbers, $\gamma \overset{def.}{=} \begin{bmatrix} 1 & 2\pi \\ 0 & 1 \end{bmatrix}$, where the linear fractional action of $SL(2, \mathbb{R})$ on H^{+} is restricted to Γ:

$$\begin{bmatrix} 1 & 2\pi n \\ 0 & 1 \end{bmatrix} \cdot (x,y) \overset{def.}{=} (x+2\pi n, y), n \in \mathbb{Z} \tag{8}$$

which by (5) is consistent with the above Schwarzschild periodicity: $(x,y) \sim (x+2\pi n, y)$.

Construction of the map $\tilde{\Phi}: T_a^{+} \to X_\Gamma$; the main observation

The tractroid T_a of radius $a > o$ of interest is the surface of revolution about the y-axis of the *tractrix curve* parametrized as follows:

$$x(t) \overset{def.}{=} ae^{-t/a}, y(t) \overset{def.}{=} a\log\left(e^{t/a} + \sqrt{e^{2t/a}-1}\right) - ae^{-t/a}\sqrt{e^{2t/a}-1} \tag{9}$$

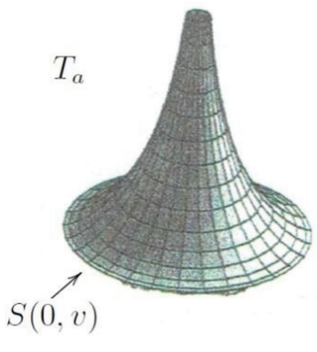
for $t \geq o$. T_a is therefore the set of points $S(u,v)$ in $\mathbb{R}^3$ given by

$$S(u,v) \overset{def.}{=} (x(u)\cos v, x(u)\sin v, y(u)) \overset{def.}{=} (ae^{-u/a}\cos v, ae^{-u/a}\sin v, S(u)),$$
$$S(u) \overset{def.}{=} y(u) \overset{def.}{=} a\log\left(e^{\frac{u}{a}} + \sqrt{e^{2u/a}-1}\right) - ae^{-u/a}\sqrt{e^{2u/a}-1} \tag{10}$$

for $(u,v) \in \mathbb{R}^2$. Since $S(0,v) = (a\cos v, a\sin v, 0)$ (as $S(0) = 0$),

$$T_a^{+} \overset{def.}{=} \{S(u,v) \in T_a \mid u > 0\} \tag{11}$$

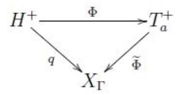
is T_a minus points on the boundary circle $S(o,v)$, as mentioned in the introduction.



Let $q: H^{+} \to X_\Gamma$ denote the quotient map that takes (x,y) to its Γ-orbit $\widetilde{(x,y)}$ in (8) and define $\Phi: H^{+} \to T_a^{+}$ by

$$\Phi(x,y) \overset{def.}{=} S\left(\log\left(\frac{y}{a}+1\right), x\right) \tag{12}$$

where we note that since $y, a > 0$, $u = \log\left(\frac{y}{a}+1\right) > 0 \Rightarrow$ indeed $\Phi(x,y) \in T_a^{+}$ by (11). Then $\tilde{\Phi}: T_a^{+} \to X_\Gamma$ is defined by the commutativity of the diagram



that is $\tilde{\Phi}\, S((u,v)) \overset{def.}{=} q\left(v, a\left(e^{u}-1\right)\right)$

(13)

for $u > 0$. For $\widetilde{(x,y)} = q(x,y)$ in X_Γ and $u = \log\left(\frac{y}{a}+1\right) > 0$ again, $a\left(e^{u}-1\right) = a\left(\frac{y}{a}+1-1\right) = y \Rightarrow p = S(u,x) \in T_a^{+}$ such that $\tilde{\Phi}(p) \overset{def.}{=} q(x,y)$, which shows that $\tilde{\Phi}$ is surjective. Finally, $\tilde{\Phi}$ is also injective and thus indeed is a bijection. Namely, if $p_j = S(u_j, v_j) \in T_a^{+}, j = 1,2$, such that $\tilde{\Phi}(p_1) = \tilde{\Phi}(p_2)$ – ie. $q\left(v_1, a\left(e^{u_1}-1\right)\right) = q\left(v_2, a\left(e^{u_2}-1\right)\right)$ (by (13)), then $v_1 = v_2 + 2\pi n$, $a\left(e^{u_1}-1\right) = a\left(e^{u_2}-1\right)$ for some $n \in \mathbb{Z}$(by (8)) $\Rightarrow u_1 = u_2$, $\cos v_1 = \cos v_2$, $\sin v_1 = \sin v_2 \Rightarrow S(u_1, v_1) = S(u_2, v_2)$ (by (10)); ie. $p_1 = p_2$.

Discussion

The BTZ vacuum (or ground state) X_Γ has a single parabolic generator γ in (7). In [4], for example, a BTZ vacuum with *two* parabolic generators is considered - in addition to other QFT matters. It would be interesting to find, also, a concrete geometric realization of the latter vacuum - or that of higher dimensional BTZ black hole vacua. One could also discuss the *naked singularity* case where $M < 0$.

Conclusion

The map $\tilde{\Phi}$ in (13) provides for a concrete, geometric, tractroid representation (or model) of the Euclidean BTZ vacuum X_Γ with Poincare metric in (6); Γ is given by (7). This result is the best possible in the sense that a general result of D.Hilbert [5] prevents the full mapping of all of T_a onto X_Γ. Our discussion proceeded at a fixed time $\tau = 0$, in which case the black hole metric (2) was reduced to the 2d spatial slice (4). One could also consider the 2d metric obtained by fixing the Schwarzschild variable ϕ in (2), and study the false vacuum decay for this 2d black hole background. Compare the interesting references [6-8], for example, where the studies therein are of a quite different focus since the word "vacuum" here simply means that we take the black hole mass M = 0 in (2). In [6], for example, the effective potential is considered for various values of the black hole mass. Also here, we need the Schwarzschild variable ϕ to be non-fixed in order to derive the Poincare metric version (6) of (4) in case M = 0, where (6) can actually be transformed to a metric on the tractroid. Thus issues regarding expectation values of quantum fluctuations and mass spectra, for example, do not arise in the present

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context, where in fact the periodicity of ϕ, moreover, which leads to equation (8), is crucial for the main construction of the bijection $\tilde{\Phi}$.

In addition to the 2d vacuum black hole-tractroid correspondence that we have constructed, there is also a 2d wormhole-catenoid correspondence. In the reference [9] a 2-dimensional section of a 3-dimensional wormhole is realized as a catenoid surface – the section is obtained by fixing a spherical polar coordinate value: $\theta = \pi/2$.

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