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Research Article

Precontinuity and applications

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Introduction

[1]In this note, a map f acting between metric (or topological) spaces is referred to be pre-continuous at a point x if, for some sequence (x_n) of points x_n different from x and converging to x, the sequence $(f(x_n))$ converges to $f(x)$ (section 2, Definition 1). We observe that this rather weak property enjoys every function with a dense graph, and a function is not pre-continuous at a point if, and only if, the respective point of its graph is isolated. In particular every additive, exponential, logarithmic, and multiplicative function is pre-continuous at every point. As a matter of fact, these functions have a stronger property, namely, they are uniformly pre-continuous (section 3, Definition 2, and Definition 3).

Another definition of continuity, using the notion of a preopen set, was introduced by Mashhour, Abd El-Monsef, and El-Deep [1] (Remark 1).

In section 4 we show that pre-continuity can be useful in solving some functional equations. Applying the property of uniform (and one-sided) pre-continuity, we determine the translative beta type functions considered in [2], the homogeneous multiplicative Cauchy quotients, and a topic leading to the Pexider equation.

Recently the family of beta-type means considered in [3,4] was applied in [5] (Remark 6).

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Pre-continuous functions

We introduce the following

Definition 1

Let $(X,d_X),(Y,d_Y)$, be metric spaces. A function $f:X\to Y$ is called pre-continuous at the point $x\in X$, if there exists a sequence (x_n), $x_n\in X\setminus\{x\}$ for $n\in\mathbb{N}$ such that $\lim_{n\to\infty}x_n=x$ and $\lim_{n\to\infty}f(x_n)=f(x)$. The function f is called pre-continuous if it is pre-continuous at every point ox X.

It is easy to construct examples of functions that are not pre-continuous. For instance, every function $f:\mathbb{R}\to\mathbb{R}$ that is increasing at 0, i.e. such that $\limsup_{x\to 0-}f(x)<f(0)<\liminf_{x\to 0+}f(x)$, is not pre-continuous at 0. On the other hand, even extremely discontinuous functions are pre-continuous. Namely, we have the following

Theorem 1

Let $(X,d_X),(Y,d_Y)$ be metric spaces and let $f:X\to Y$ be an arbitrary function.

(i) If the graph f is dense in the product metric space $X\times Y$, then f is pre-continuous.

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(ii) The function f is pre-continuous at a point $x \in X$ if and only if $(x, f(x))$ is not an isolated point of the graph f.

(iii) If f is continuous at an accumulation point X, then f is pre-continuous at the point ·

Proof. (i) Take an arbitrary point $x \in X$. The density of the graph of f there exists a sequence $(x_n, f(x_n)) \in X \quad X \times Y$ $n \in \mathbb{N}$, such that $\lim_{n\to\infty}(x_n, f(x_n)) = (x, f(x))$ in the product metric, that is such that $\lim_{n\to\infty} d_X(x_n, x) = 0$ and $\lim_{n\to\infty} d_Y(f(x_n), f(x)) = 0$, which completes the proof.

(ii) If f is pre-continuous at a point $x \in X$ then, by Definition 1, there is a sequence (x_n), $x_n x_n \in X \setminus \{x\}$ for $n \in \mathbb{N}$ such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} f(x_n) = f(x)$. Without any loss of generality, we can assume that (x_n) is one-to-one. Then $(x_n, f(x_n)) \in \quad X \times Y$, $n \in \mathbb{N}$, is a sequence of different points of the graph f converging $(x, f(x))$ in the product topology. The converse implication follows from the definition of the product topology.

We omit an easy argument for (iii).

It is well-known that the graph of every discontinuous additive function $\alpha : \mathbb{R} \to \mathbb{R}$ is dense $\mathbb{R}^2$ (as well as the graphs of discontinuous multiplicative, exponential, and logarithmic functions are dense in the suitable natural subsets $\mathbb{R}^2$).

Theorem 2

Let $I \subset \mathbb{R}$ be an interval. For an arbitrary function, $f : I \to \mathbb{R}$ the set of all points $x \in I$ that f is not pre-continuous x is at most countable.

Proof. Let $Z \subset X$ be the set of all $x \in I$ such that f is not pre-continuous at x. By Theorem 1(ii), the set $\{(x, f(x)) : x \in Z\}$ is the set of all isolated points of the graph f, that are contained $I \times \mathbb{R}$. But, clearly, the set of isolated points of any subset $I \times \mathbb{R}$ is at most countable.

Remark 1

Let $(X, \mathcal{T})$, $(Y, \mathcal{S})$ be topological spaces. In [1] a function $f : X \to Y$ is said to be pre-continuous at a point $x \in X$, if for every open set $V \in \mathcal{S}$ containing $f(x)$ there is a set $U \subset X$ such that $x \in U$, $U \subset Int(Cl(U))$ (preopeness) and $f(U) \subset V$ (see also [6]).

For obvious reasons, the notion of precontinuity proposed in Definition 1 could be called a Heine-type. We omit to discuss the mutual relations between these two concepts.

Uniform pre-continuous functions

Definition 2

Let X be a subset of a metric group G with an addition "$+$" and neutral element 0 and Y be a metric space. A function $f : X \to Y$ is called uniformly pre-continuous, if there exists a sequence $z_n \in G \setminus \{0\}$ for all $n \in \mathbb{N}$, with $\lim_{n\to\infty} z_n = 0$ such that $x + \quad z_n \in X$, for all $x \in X$, $n \in \mathbb{N}$, and $\lim_{n\to\infty} f(x + z_n) = f(x)$.

Remark 2

Let X be a metric group with an addition "$+$" and neutral element 0, Y be a metric space, and $f : X \to Y$ be an additive function, i.e.

$$f(x+y) = f(x) + f(y), \quad x, y \in X.$$

The following two conditions are equivalent

(i) f is pre-continuous at a point;

(ii) f is uniformly pre-continuous .

Proof. To prove (i) $\Longrightarrow$ (ii) assume that for some $x_0 \in X$ there is $x_n \in X \setminus \{x_0\}$ for all $n \in \mathbb{N}$ such that $\lim_{n\to\infty} x_n = x_0$ $\lim_{n\to\infty} f(x_n) = f(x_0)$. Putting $z_n := x_n - x_0$ we have $z_n \in X \setminus \{0\}$ for every $n \in \mathbb{N}$ and $\lim_{n\to\infty} z_n = 0$. Hence, for arbitrary $x \in X$, making use of the additivity of f and its oddness, we

$$\lim_{n\to\infty} f(x+z_n) = \lim_{n\to\infty}[f(x) + f(z_n)] = f(x) + \lim_{n\to\infty} f(z_n)$$

$$= f(x) + \lim_{n\to\infty} f(x_n - x_0) = f(x) + \lim_{n\to\infty}[f(x_n) - f(x_0)]$$

$$= f(x) + [f(x_0) - f(x_0)] = f(x),$$

Which proves (ii). The converse implication is trivial.

Remark 3

In the case of functions of real variable we define a uniformly right-pre-continuous (left-pre-continuous) function, postulating

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that the respective zero sequences (z_n) are positive (resp. negative). In this case, the oddness of the additive function implies that the above result remains true if (i) is replaced by " f is left- or right-pre-continuous at a point".

Corollary 1

Let X be a metric group with an addition "$+$" and a neutral element 0. If a function $f: X \to \mathbb{R}$ is additive and discontinuous at a point, then its graph is dense in the product metric space $X \times \mathbb{R}$.

Proof. Assume that $f: X \to \mathbb{R}$ is additive and discontinuous at a point. Of course, f is discontinuous 0 (see, for instance [7]). Since $f(0)=0$ there is a sequence (z_n) with $\lim_{n\to\infty} z_n = 0$ such that either $\lim_{n\to\infty} f(z_n)$ is a finite nonzero real number or $\lim_{n\to\infty} f(z_n) = \infty$.

In the first case, we can assume that

$$\lim_{n\to\infty} f(z_n) = 1$$

If the second case holds, choosing a sequence of real rational numbers (r_n), $r_n \neq 0$ for all $n \in \mathbb{N}$, such that

$$\lim_{n\to\infty} \frac{f(z_n)}{r_n} = 1,$$

and putting

$$w_n := \frac{z_n}{r_n}, \qquad n \in \mathbb{N},$$

we have

$$\lim_{n\to\infty} w_n = 0,$$

and, making use of the rational homogeneity of f (Aczél [8] p. 32, Kuczma [7] p. 121, Theorem 1),

$$\lim_{n\to\infty} f(w_n) = \lim_{n\to\infty} f\left(\frac{z_n}{r_n}\right) = \lim_{n\to\infty} \frac{f(z_n)}{r_n} = 1.$$

Thus, in both possible cases, there exists a sequence (z_n) $\lim_{n\to\infty} z_n = 0$ such that

$$\lim_{n\to\infty} f(z_n) = 1.$$

Hence, by the rational homogeneity of f, every rational number $r \in \mathbb{Q}$, we have

$$\lim_{n\to\infty} f(rz_n) = r \lim_{n\to\infty} f(z_n) = r,$$

Which implies that every point $\{0\} \times \mathbb{R} \{(0,t) : 0 \in X \wedge t \in \mathbb{R}\}$ is an accumulation point of the graph f. Through the additivity f we have

$$f(x + rz_n) = f(x) + f(rz_n), \qquad x \in X, r \in \mathbb{Q}$$

so, for every point $x \in X$, the set $\{x\} \times \mathbb{R}$ is contained in the closure of the graph f. This completes the proof.

Definition 3

Let X be a subset of a metric group G with a multiplication "$\cdot$" and neutral element 1 and Y be a metric space. A function $f: X \to Y$ is called uniformly pre-continuous, if there exists a sequence $z_n \in G \setminus \{1\}$ for all $n \in \mathbb{N}$, with $\lim_{n\to\infty} z_n = 1$ such that for all $x \in X$, $n \in \mathbb{N}$, we have $x \cdot z_n \in X$, and $\lim_{n\to\infty} f(x \cdot z_n) = f(x)$.

Remark 4

Here, in the case of functions of real variable we define uniformly right-pre-continuous (left-pre-continuous) functions, postulating that the respective zero sequences (z_n) are such that $z_n > 0$ (resp. $z_n < 0$) for all $n \in \mathbb{N}$.

Theorem 3

Every additive function $\alpha: \mathbb{R} \to \mathbb{R}$ is uniformly pre-continuous.

Every additive function $\alpha: (0,\infty) \to \mathbb{R}$ is right-uniformly pre-continuous.

Indeed, assume that $\alpha: \mathbb{R} \to \mathbb{R}$ is additive and take $z_n = \frac{1}{n}$ for $n \in \mathbb{N}$. As every additive function is rationally homogeneous, we have for every $x \in \mathbb{R}$ and $n \in \mathbb{N}$,

$$\alpha\left(x + \frac{1}{n}\right) = \alpha(x) + \alpha\left(\frac{1}{n}\right) = \alpha(x) + \frac{1}{n}\alpha(1),$$

Whence $\lim_{n\to\to\infty} \alpha\left(x + \frac{1}{n}\right) = \alpha(x)$. Since $\mathbb{R}$ the addition is a group of the neutral element 0, and the euclidean topology, satisfies the conditions of Definition 2, the function α is uniform and pre-continuous $\mathbb{R}$.

The argument for the second result is analogous.

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Corollary 2

Every exponential function $f:\mathbb{R}\to(0,\infty)$, *i.e. such that*

$$f(x+y)=f(x)f(y),\quad x,y\in\mathbb{R},$$

is uniformly pre-continuous (in the additive group $\mathbb{R}$).

Proof. It $f:\mathbb{R}\to(0,\infty)$ is exponential then $\alpha:=\log\circ f$ is additive in $\mathbb{R}$, and $f=\exp\circ\alpha$. Thus, for all $x\in\mathbb{R}$ and $n\in\mathbb{N}$, by the additivity of α, similarly as above, we have

$$f\left(x+\frac{1}{n}\right)=f(x)f\left(\frac{1}{n}\right)=f(x)\exp\left(\alpha\left(\frac{1}{n}\right)\right)=f(x)\exp\frac{\alpha(1)}{n}=e^{\frac{\alpha(1)}{n}}f(x),$$

so $\lim_{n\to\to\infty}f\left(x+\frac{1}{n}\right)=f(x)$. Using Definition 2 with $X=\mathbb{R}$ and $z_n=\frac{1}{n}$ for $n\in\mathbb{N}$, we get the result.

Corollary 3

Every logarithmic function $f:(0,\infty)\to\mathbb{R}$, *i.e. such that*

$$f(xy)=f(x)+f(y),\quad x,y\in(0,\infty).$$

is uniformly pre-continuous (in the multiplicative group $(0,\infty)$).

Proof. The interval $(0,\infty)$ with the multiplication, neutral element 1, and the euclidean topology, satisfies the conditions of Definition 2. It $f:(0,\infty)\to\mathbb{R}$ is logarithmic then $\alpha:\mathbb{R}\to\mathbb{R}$ defined by $\alpha:=f\circ\exp$ is additive, and $f=\alpha\circ\log$. Taking $z_n:=\exp\left(\frac{1}{n}\right)$ we have $\lim_{n\to\infty}z_n=1$ and for all $x>0$ and $n\in\mathbb{N}$,

$$f(x\cdot z_n)=f(x)+f(z_n)=f(x)+\alpha(\log z_n)=f(x)+\alpha\left(\log e^{\frac{1}{n}}\right)$$

$$=f(x)+\alpha\left(\frac{1}{n}\right)=f(x)+\frac{\alpha(1)}{n},$$

whence $\lim_{n\to\infty}f(x\cdot z_n)=f(x)$, which, in view of Definition 3, shows that the function f is uniformly pre-continuous.

Corollary 4

Every multiplicative function $f:(0,\infty)\to(0,\infty)$, *i.e. such that*

$$f(xy)=f(x)f(y),\quad x,y\in(0,\infty),$$

is uniformly pre-continuous (in the multiplicative group $((0,\infty),\cdot)$).

Proof. The function $\alpha:\mathbb{R}\to\mathbb{R}$ defined by $\alpha:=\log\circ f\circ\exp$ is additive, and $f=\exp\circ\alpha\circ\log$ we can argue similarly as in the proof of Corollary 3.

Examples of applications

To illustrate the possible advantages of the introduced notions we begin with the following

Proposition 1

The functions $f,g:(0,\infty)\to\mathbb{R}$ *satisfy the equation*

$$f(x+y)+g(z)=f(z+y)+g(x),\quad x,y,z\in(0,\infty), \tag{1}$$

and f is uniformly right-pre-continuous, if and only if

$$f=\alpha+b,\qquad g=\alpha+c$$

for some additive function $\alpha:(0,\infty)\to\mathbb{R}$ and $b,c\in\mathbb{R}$.

Proof. Assume that f,g satisfy this equation (1) and f is uniformly right-pre-continuous. Writing this in the form

$$f(x+y)-g(x)=f(z+y)-g(z),\quad x,y,z\in(0,\infty),$$

We see that the difference $f(x+y)-g(x)$ does not depend on x, so the function $h:(0,\infty)\to\mathbb{R}$ given by

$$h(y):=f(x+y)-g(x)$$

is well defined and, consequently, the Pexider functional equation

$$f(x+y)=g(x)+h(y),\quad x,y\in(0,\infty), \tag{2}$$

is satisfied. In view of Definition 1 (see also Remark 2), there exists a positive sequence (z_n) tending to 0 such that for every $x>0$,

$$\lim_{n\to\infty}f(x+z_n)=f(x).$$

Setting $y=z_n$ in (2) we have, for every $x>0$,

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$$f(x+z_n)=g(x)+h(z_n), \quad n\in\mathbb{N},$$

and letting $n\to\infty$, we obtain conclude that

$$f(x)=g(x)+h_0, \quad x\in(0,\infty), \tag{3}$$

where

$$h_0 := \lim_{n\to\infty} h(z_n)$$

Exists and does not depend on x.

Similarly, taking $x := z_n$ in (2), we have

$$f(z_n+y)=g(z_n)+h(y), \ n\in\mathbb{N},$$

and letting $n\to\infty$, us obtain

$$f(y)=g_0+h(y), \quad y>0, \tag{4}$$

Where

$$g_0 := \lim_{n\to\infty} g(z_n)$$

is a real constant. From (2), (3), and (4), setting

$$b := g_0+h_0,$$

we get

$$f(x+y)-b=[f(x)-b]+[f(y)-b], \ x,y\in(0,\infty),$$

which shows that $\alpha := f-b$ is an additive function, and

$$f=\alpha+b.$$

Setting this function into equation (1) gives

$$g(x)-\alpha(x)=g(z)-\alpha(z), \quad x,z\in(0,\infty),$$

that is $g-\alpha=c$ for some real c. Thus

$$g(x)=\alpha(x)+c, \ x\in(0,\infty).$$

The converse implication follows from the fact that α is uniformly right-pre-continuous (Theorem 2).

For a function $f:(1,\infty)\to(0,\infty)$ define the bivariate function $P_f:(1,\infty)^2\to(0,\infty)$ by

$$P_f(x,y) := \frac{f(x)f(y)}{f(xy)}, \quad x,y>1.$$

Proposition 2

Let $f:(1,\infty)\to(0,\infty)$ be uniformly right-pre-continuous and $m:(1,\infty)\to(0,\infty)$ be an arbitrary function.

Then the following conditions are equivalent

(i) the function P_f is m-homogeneous, i.e.

$$P_f(tx,ty)=m(t)P_f(x,y), \ t,x,y>1; \tag{5}$$

ii) the function $m\equiv 1$ and there is $b>0$ such that the function $\frac{f}{b}$ is multiplicative, i.e.

$$bf(xy)=f(x)f(y), \ x,y>1.$$

Proof. Assume (i). Then for all $s,t,x,y>1$ we have

$$m(st)=\frac{P_f(stx,sty)}{P_f(x,y)}=\frac{P_f(s(tx),s(ty))}{P_f(tx,ty)}\frac{P_f(tx,ty)}{P_f(x,y)}=m(s)m(t),$$

so m is multiplicative.

The interval $(1,\infty)$ is a subset of the multiplicative group $((0,\infty),\cdot)$ with neutral element 1. Let (z_n) be a sequence satisfying the conditions of Definition 3 of uniform right-precontinuity of f in $(1,\infty)$; in particular $z_n>1$ for $n\in\mathbb{N}$ and $\lim_{n\to\infty} z_n=1$. Using the definition of P_f and setting $y := z_n$ in (5) we have

$$\frac{f(tx)f(tz_n)}{f(t^2xz_n)}=m(t)\frac{f(x)f(z_n)}{f(xz_n)}, \quad t,x>1, n\in\mathbb{N}.$$

Letting $n\to\infty$ us conclude that

$$b := \lim_{n\to\infty} f(z_n)$$

exists, is positive, finite, and

$$\frac{f(tx)f(t)}{f(t^2x)}=bm(t), \quad t,x>1.$$

Thus $\frac{f(tx)}{f(t^2x)}$ does not depend on x. So, replacing hereby x, and setting

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$$g(t) := \frac{f(x)f(t)}{f(tx)m(t)}, \quad t > 1,$$

we get

$$f(tx) = g(t)f(x), \ t, x > 1. \tag{6}$$

Taking here $x = z_n$, $n \in \mathbb{N}$, as above, we have

$$f(tz_n) = g(t)f(z_n), t > 1, n \in \mathbb{N}.$$

Letting $n \to \infty$, the assumed precontinuity of f gives

$$f(t) = bg(t), \ t > 1, \tag{7}$$

Hence, making use of (6), we have

$$g(tx) = g(t)g(x), \ t, x > 1, (8)$$

that is *g* is multiplicative.

Applying in turn (5), the definition of P_f, (7) and (8) we get, for all $t, x, y > 1$,

$$m(t) = \frac{P_f(tx, ty)}{P_f(x, y)} = \frac{f(tx)f(ty)f(xy)}{f(t^2xy)f(x)f(y)}$$

$$= \frac{g(tx)g(ty)g(xy)}{g(t^2xy)g(x)g(y)} = \frac{[g(t)]^2[g(x)]^2[g(y)]^2}{[g(t)]^2[g(x)]^2[g(y)]^2}$$

$$= 1$$

which completes the proof of (ii).

The implication $(ii) \Rightarrow (i)$ is obvious·

Remark 5

Of course, the counterpart of the above result for function $f:(0,1) \to (0,\infty)$ *also holds true.*

Let $f:(0,\infty) \to (0,\infty)$ be an arbitrary function. The two-variable functions $B_f:(0,\infty)^2 \to (0,\infty)$ given by

$$B_f(x, y) := \frac{f(x)f(y)}{f(x+y)}, \ x, y > 0,$$

is called a *beta-type function, and f* is referred to as its generator *([2]).*

Remark 6

Note that Barczy and Burai [5] have derived strong laws of large numbers and central limit theorems, among others, for a new type family of beta-type means considered in [3] and [4].

A function $F:(0,\infty)^2 \to \mathbb{R}$ is called *translative* with respect to a function $\alpha:(0,\infty) \to \mathbb{R}$, if

Remark 7

If F is translative with respect to α *then* α *is an additive function. If moreover F nonnegative, then there is a* $a \in \mathbb{R}$ $a \geq 0$ *such that* $\alpha(t) = at$ *for all* $t > 0$.

Proof. Indeed, for all $x, y, s, t \in (0,\infty)$ we have

$$F(x+s+t, y+s+t) = F((x+s)+t, (y+s)+t) =$$

$$F(x+s, y+s) + \alpha(t) = F(x, y) + \alpha(s) + \alpha(t),$$

and

$$F(x+s+t, y+s+t) = F(x, y) + \alpha(s+t),$$

whence $\alpha(s+t) = \alpha(s) + \alpha(t)$, so α is additive in $(0,\infty)$.

From the transitivity of F and the just proved additivity of α we have, for all $x, y, t > 0$ and $n \in \mathbb{N}$,

$$F(x+nt, y+nt) = F(x, y) + \alpha(nt) = F(x, y) + n\alpha(t).$$

Clearly, this equality and the assumed nonnegativity F exclude existence $t > 0$ $\alpha(t) < 0$.

Proposition 3

Let $f:(0,\infty) \to (0,\infty)$ *be a (right) uniformly pre-continuous function and* $\alpha:(0,\infty) \to \mathbb{R}$ *be given functions. The following conditions are equivalent:*

(i) the beta-type function $B_f:(0,\infty)^2 \to (0,\infty)$ is translative with respect to the function α;

(ii) $\alpha \equiv 0$ and, for some $c > 0$, the function $\frac{f}{c}$ is an exponential function, i.e.

$$cf(x+y) = f(x)f(y), \ x, y > 0.$$

Proof. Assume (i). In view of Remark 4, there is a real

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number $a \ge 0$ such that $\alpha(t) = at$ for all $t > 0$ and from the assumed transitivity B_f, we have

$$\frac{f(x+t)f(y+t)}{f(x+y+2t)} = \frac{f(x)f(y)}{f(x+y)} + at, \quad x,y,t > 0.$$

Hence, for all $x > y > 0$, and $s > 0$,

$$\frac{f(x)f(y+s)}{f(x+y+s)} = \frac{f((x-y)+y)f(s+y)}{f([(x-y)+y]+(s+y))} = \frac{f(x-y)f(s)}{f(x-y+s)} + ay.$$

Setting here $s = z_n$, where (z_n) is a sequence such that is $z_n > 0$ all $n \in \mathbb{N}$, $\lim_{n\to\infty} z_n = 0$, satisfying the condition of the uniform right-precontinuity, we have

$$\frac{f(x)f(y+z_n)}{f(x+y+z_n)} = \frac{f(x-y)f(z_n)}{f(x-y+z_n)} + ay, \quad n \in \mathbb{N}, 0 < y < x.$$

Letting $n \to \infty$, and making use of the right continuity f, we conclude that the limit

$$b := \lim_{n\to\infty} f(z_n) \tag{9}$$

exists, is nonnegative, finite and

$$\frac{f(x)f(y)}{f(x+y)} = b + ay, \; 0 < y < x. \tag{10}$$

or, equivalently, that

$$\frac{f(x)f(y)}{f(x+y)} = b + a\min(x,y), \quad x,y > 0.$$

For arbitrary $x, z > 0$, choosing positive y such that $y < x$ and $y < z$, we hence get

$$\frac{f(x)f(y)}{f(x+y)} = b + ay = \frac{f(z)f(y)}{f(z+y)},$$

whence

$$\frac{f(z+y)}{f(z)} = \frac{f(x+y)}{f(x)}.$$

It follows that the function $g:(0,\infty) \to (0,\infty)$

$$g(y) := \frac{f(x+y)}{f(x)}, \quad y > 0,$$

is well defined. Since f,g are continuous and satisfy the Pexider functional equation

$$f(x+y) = f(x)g(y), \quad x,y > 0. \tag{11}$$

By the symmetry of the left-hand-side x and y we have

$$f(x+y) = f(y)g(x), \quad x,y > 0.$$

Setting here $y = z_n$, where the sequence $y = z_n$ is chosen above, we have

$$f(x+z_n) = f(z_n)g(x), \quad n \in \mathbb{N}, x,y > 0.$$

Letting here $n \to \infty$, and using (9), we get

$$f(x) = bg(x), \quad x > 0, \tag{12}$$

which implies that $b \ne 0$. Hence, using (11), we obtain

$$g(x+y) = g(x)g(y), \quad x,y > 0,$$

which means that g is an exponential function. From (10) we get $a = 0$, and using (12) we conclude (ii).

The implication $(ii) \Rightarrow (i)$ is obvious.

Proposition 4

If the functions $f, g, h : (0,\infty) \to \mathbb{R}$ *satisfy the equation*

$$f(x+y) = g(x) + h(y), \quad x,y \in (0,\infty), \tag{13}$$

then

$$f = \alpha + b + c, \quad g = \alpha + c, \quad h = \alpha + b$$

for some additive function $\alpha : (0,\infty) \to \mathbb{R}$ and $b, c \in \mathbb{R}$.

Proof. From (13), making use of the commutativity of addition, we have for all $x, y \in (0,\infty)$

$$g(x) + h(y) = f(x+y) = f(y+x) = g(y) + h(x),$$

whence, for all $x, y \in (0,\infty)$,

$$h(x) = g(x) + h(y) - g(y).$$

Choosing arbitrarily $y = y_0 > 0$, we get

$$h(x) = g(x) + h(y_0) - g(y_0), \quad x > 0. \tag{14}$$

Setting this into (13) we get

$$f(x+y) - [g(y_0) + h(y_0)] = [g(x) - g(y_0)] + [g(y) - g(y_0)], \; x, y > 0,$$

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whence, setting

$$\bar{f}(x) := f(x) - \left[g(y_0) + h(y_0)\right], \ \bar{g}(x) := g(x) - g(y_0), \ x > 0, \tag{15}$$

we obtain

$$\bar{f}(x+y) = \bar{g}(x) + \bar{g}(y), \ x, y > 0. \tag{16}$$

Hence, by induction we get

$$\bar{f}(x_1 + \ldots + x_n) = \bar{g}(x_1) + \ldots + \bar{g}(x_n), \quad n \in \mathbb{N}, n \geq 2; x_1, \ldots, x_n > 0,$$

Whence

$$\bar{f}(nx) = n\bar{g}(x), \ n \in \mathbb{N}, n \geq 2; x > 0.$$

Replacing hereby $\frac{x}{n}$, we get

$$\bar{g}\left(\frac{x}{n}\right) = \frac{\bar{f}(x)}{n}, \ n \in \mathbb{N}, n \geq 2; x > 0,$$

which implies that

$$\lim_{n\to\infty} \bar{g}\left(\frac{1}{n}\right) = 0.$$

Now (16) implies that $\bar{g}$ is uniformly pre-continuous $z_n = \frac{1}{n}$. Of course, (16), $\bar{f}$ is uniformly pre-continuous, and from (14) and (15) it follows that f, g, h are uniformly pre-continuous with the same sequence $z_n = \frac{1}{n}$.

In view of Definition 1 (see also Remark 2), there exists a positive sequence (z_n) tending to 0 such that for every $x > 0$,

$$\lim_{n\to\infty} f(x + z_n) = f(x).$$

Setting $y = z_n$ in (13) we have, for every $x > 0$,

$$f(x + z_n) = g(x) + h(z_n), n \in \mathbb{N},$$

and letting $n \to \infty$, we obtain conclude that

$$f(x) = g(x) + b, \ x \in (0, \infty), \tag{17}$$

Where

$$b := \lim_{n\to\infty} h(z_n)$$

exists and does not depend on x.

Similarly, taking $x := z_n$ in (13), we have

$$f(z_n + y) = g(z_n) + h(y), \ n \in \mathbb{N},$$

and letting $n \to \infty$, us obtain

$$f(y) = c + h(y), \ y > 0, \tag{18}$$

where

$$c := \lim_{n\to\infty} g(z_n)$$

is a real constant. From (13), (17), and (18), setting

$$a := b + c,$$

we get

$$f(x+y) - (b+c) = \left[f(x) - (b+c)\right] + \left[f(y) - (b+c)\right], \ x, y \in (0, \infty),$$

which shows that $\alpha := f - (b+c)$ is an additive function, and

$$f = \alpha + (b+c).$$

Hence, from (17) we get

$$g(x) = \alpha(x) + c, \ x \in (0, \infty),$$

and from (18),

$$h(x) = \alpha(x) + b, \ x \in (0, \infty),$$

Which completes the proof.

Final Remark

Following Azad [9] one could try to consider the fuzzy versions of precontinuity.

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